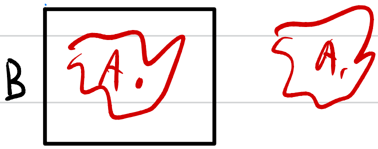
Descriptive Set Theory
Lecture 21

Analytic Separation (Luzin). If $A_0, A_1 \subseteq X$ Polish are disjoint analytic sets, then $\exists\, B \subseteq X$ Borel separating them, i.e. $B \supseteq A_0$ and $B^c \supseteq A_1$.



Proof. Call sets A and B *Borel-separable* if $\exists$ Borel sets R s.t. $A \subseteq R$ and $B \subseteq R^c$. Note that for a fixed set A, sets that are Borel-separable from A form a σ-ideal, indeed if each B_n is separated from A by a Borel set R_n, i.e. $R_n^c \supseteq A$ and $R_n \supseteq B_n$, then $\bigcup_n R_n \supseteq \bigcup_n B_n$ and $\left(\bigcup_n R_n\right)^c \supseteq A$. Moreover:

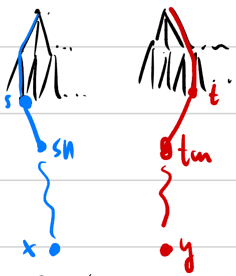
Claim. For sets (P_n) and (Q_m), if each P_n is Borel-separable from each Q_m, then $\bigcup_n P_n$ is Borel-separable from $\bigcup_m Q_m$.

Proof. Fix P_n. Then each Q_m is Borel-sep. from P_n, hence so is $\bigcup_m Q_m$. Then each P_n is Borel-sep. from $\bigcup_m Q_m$, hence so is $\bigcup_n P_n$. $\square$

By Hausdorffness, distinct points x and y are open-separable, i.e.

have disjoint open neighbourhoods. We will show that if two disjoint analytic sets A and B are not Borel-sep., then some $a \in A$ and $b \in B$ are not open-separable, a contradiction.

Let $f : \mathbb{N}^{\mathbb{N}} \twoheadrightarrow A$ and $g : \mathbb{N}^{\mathbb{N}} \twoheadrightarrow B$ be continuous surjections, and suppose that A and B are not Borel-sep. For each $s \in \mathbb{N}^{<\mathbb{N}}$, put $A_s := f([s])$ and $B_s := g([s])$. What the claim above says is that if A_s and B_t are not Borel-sep., for $|s| = |t|$, then $\exists$ A_{sn} and B_{tm} that are still not Borel-sep. Thus iterating this, we follow the non-sep. pair of branches in the tree $\mathbb{N}^{<\mathbb{N}}$, building $x, y \in \mathbb{N}^{\mathbb{N}}$ s.t. $\forall n$ $A_{x|n}$ and $B_{y|n}$ are not Borel-sep.



Let $a := f(x)$ and $b := g(y)$. Let $U \ni a$ and $V \ni b$ be disjoint open sets. By continuity of f and g, $\exists n$ s.t. $A_{x|n} := f([x|n]) \subseteq U$ and $B_{y|n} := g([y|n]) \subseteq V$, so U separates $A_{x|n}$ and $B_{y|n}$, a contradiction. $\square$

Cor. Any ctbl disjoint collection (A_n) of analytic sets can be Borel-separated, i.e. $\exists$ pairwise disjoint Borel sets $B_n \supseteq A_n$.

Proof. Exercise. $\square$

Borel graph theorem. For Polish sp. X, Y, and a function $f: X \to Y$, TFAE, where $G_f := \operatorname{graph}(f) := \{(x,y) \in X \times Y : f(x) = y\}$:

(1) f is a Borel function, i.e. $f^{-1}(\text{Borel})$ is Borel.

(2) G_f is Borel.

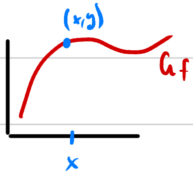
(3) G_f is analytic.

Proof. (1) $\Rightarrow$ (2). Fix a ctbl open basis (V_n) for Y. For any $(x,y) \in X \times Y$,

$$(x,y) \in G_f \iff f(x) = y \iff x \in f^{-1}(y) \iff \forall n \, (y \in V_n \Rightarrow x \in f^{-1}(V_n))$$

$$\iff \forall n \, \big(\underbrace{y \notin V_n}_{\text{closed}} \text{ or } \underbrace{x \in f^{-1}(V_n)}_{\text{Borel}} \big).$$

(3) $\Rightarrow$ (1). Fix an open $V \subseteq Y$ and show that $f^{-1}(V)$ is Borel.



For each $x \in X$,

$$x \in f^{-1}(V) \iff \overbrace{\exists y \in Y \, \big(\overbrace{(x,y) \in G_f}^{\text{analytic}} \text{ and } \overbrace{y \in V}^{\text{open}} \big)}^{\text{analytic}}$$

$$\iff \forall y \in Y \, \big((x,y) \in G_f \Rightarrow y \in V \big)$$

$$\iff \underbrace{\forall y \in Y \, \big(\underbrace{\underbrace{(x,y) \notin G_f}_{\text{coanalytic}} \text{ or } \underbrace{y \in V}_{\text{open}}}_{\text{coanalytic}} \big)}_{\text{coanalytic}}.$$

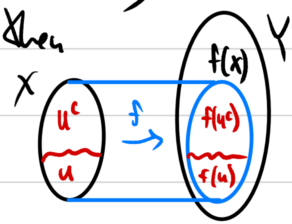
Thus, $f^{-1}(V)$ is in $\Delta^1_1(X)$, hence is Borel. $\square$

Closure of Borel set under small-to-one Borel functions.

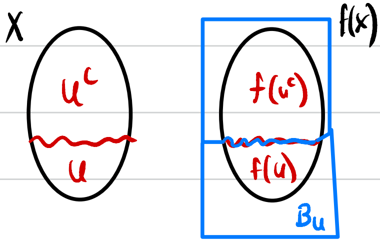
As we saw, Borel sets are not closed under continuous/Borel images. However, it turns out that if the preimage of each point is small (e.g. ≤ 1 pt, ctbl, compact, σ-compact) then the image of Borel sets are still Borel.
The first theorem of this kind is:

1-1 images (Luzin–Souslin). For any Polish $(X,\mathcal{T}_X),(Y,\mathcal{T}_Y)$, any 1-to-1 Borel function $f: X \to Y$ maps Borel sets to Borel sets.

Proof (R. Chen, 2017). We may assume f is continuous, by refining the top on X, and it is enough to prove that $f(X)$ is Borel, for the same reason (for any Borel $B \subseteq X$, make it clopen and consider the function $f|_B$ instead).
Assuming $f: X \to Y$ is an embedding, $f: X \to f(X)$ is a homeo, then



$f(X)$ is also Polish, hence G_δ. Thus our goal is to refine the Polish top. on both X and Y, keeping the same Borel sets, but making f into an embedding, hence then f(X) would be G_δ in the finer top, hence Borel in the original top of Y. To this end, fix a ctbl basis $\mathcal{U}$ for X and to make f into an open map, it's enough to turn $f(U)$ open relative to $f(X)$, for each $U \in \mathcal{U}$. $f(U)$ is analytic, but note that $f(U^c)$ is also analytic and disjoint by injectivity of f. By analytic separation, $\exists$ Borel $B_U \subseteq Y$ s.t. $f(U) \supseteq B_U$ and $f(U^c) \supseteq B_U^c$. Let $\mathcal{T}_Y'$ be the refinement



of the top. on Y that is Polish with the same Borel sets but makes each B_U clopen, for all $U \in \mathcal{U}$. This makes $f(U)$ clopen relative to $f(X)$ because $f(U^c) = f(U)^c \cap f(X)$, so $f(X) \cap B_U^c = f(U^c)$.

Now $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y')$ is an open map but it's no longer continuous, although graph(f) is still closed in $(X \times Y, \mathcal{T}_X \times \mathcal{T}_Y')$. Therefore, the top $\mathcal{T}_X' := \mathcal{T}_X + f^{-1}(\mathcal{T}_Y')$ is still Polish by the black magic proven last time. Now, $f : (X, \mathcal{T}_X') \to (Y, \mathcal{T}_Y')$ is continuous, and it is still open because the images of new open sets are open by def, indeed, $f(f^{-1}(V)) = V$ for any $V \in \mathcal{T}_Y'$.

Hence f is an embedding, which is what we wanted. $\square$

Cor. Let $(X, \mathcal{T}_X)$ be a Polish space and let $\mathcal{T}_X'$ be a Polish refinement. Then $\mathcal{T}_X' \subseteq \mathcal{B}(X, \mathcal{T}_X)$.

Proof. The identity map $id : (X, \mathcal{T}_X') \to (X, \mathcal{T}_X)$ is continuous, hence Borel, and 1-1, so it maps Borel sets to Borel, in particular, it maps any $\mathcal{T}_X'$-open set to a Borel set. $\square$

Cor. Any injective Borel map between Polish spaces is a Borel embedding (i.e. maps Borel to Borel).

Recall that each Polish space is a 1-1 continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$. This characterizes all Borel sets.

Cor. For a Polish space X, Borel subsets of X are exactly the 1-1 continuous images of closed subsets of $\mathbb{N}^{\mathbb{N}}$.

Proof. Let $B \subseteq X$ be Borel, then by change of top, B is clopen, hence itself Polish hence a 1-1 continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, and this function is "even more" continuous in the original top. of X.
The converse is by the Luzin–Souslin theorem. $\square$